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Products in the category of locales: which properties are preserved?

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Abstract

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In this paper we are going to study the category of finitely regular locales. We shall show that any product of finitely regular locales with some covering property has this property as well.
As a partial result we have that any product of compact (paracompact) locales is compact (paracompact).

Introduction

This paper is the first of two in which we aim to investigate the category of finitely regular locales together with questions concerning the preservation of some covering properties (compactness, paracompactness etc.).

The paper is organized in the following way. In Section 1 we recall the basic notions concerning the theory of locales and topology. In Section 2 we discuss the concept of the finite regularity in locales and its connection with some covering and separation lattice-topological axioms (paracompactness, regularity). The general cover theorem is the main result of Section 3. It shows that under some restricted assumptions certain covering properties of locales are preserved in products. The first one is the finite regularity mentioned above. The second assumption corresponds in fact to the conditions (i)–(iii) of the paper [2]. Section 4 is devoted to locale-theoretic and topological applications of Section 3.

The results of Kovár (see [5]) have been a source of inspiration for the problems concerning finite regularity of this paper.

1. Preliminaries

The basic reference for the present text is the classic book by Johnstone [4], where the interested reader can find unexplained terms and notation concerning the subject. For facts concerning topology in general we refer to [2, 7]. Our terminology and notation agree with the book [4] of Johnstone and with the papers [1, 8].

We now review some terminology from elementary topology and locales.

Recall that for F, G sets, F a finite subset of G we write $F \subseteq\subseteq G$. If X is a topological space we put $\tau(X)$ for its frame of open sets and $\Omega(X)$ for the corresponding locale. For a locale L we denote by $\tau(L)$ the frame we are dealing with. Locale morphisms are said to be continuous maps. A nucleus j on L is said to be:

(i) dense if $j(a) = 0$ implies $a = 0$,
(ii) codense if $j(a) = 1$ implies $a = 1$,
(iii) finitary if $S \subseteq L_j$, S directed implies $j(\bigvee S) = \bigvee S$.

Next, we shall describe the construction of products in the category $\mathscr{L}oc$ of locales.

Let L_γ, $\gamma \in \Gamma$, be a family of locales and write B' for the set-theoretical product of the $\tau(L_\gamma)$. Clearly, B' is a frame and the projections $\pi_\gamma : B' \to L_\gamma$ are frame morphisms. We put $B = \{x' \in B' : \pi_\gamma(x') = 1$ for all but finitely many $\gamma \in \Gamma\}$. Clearly, B is a meet semilattice. We define $Z = \downarrow B$ to be the frame of all lower sets of B. We put $M = \{x' \in B : \pi_\gamma(x') = 0$ for some $\gamma \in \Gamma\}$ and $Q = \{W \subseteq B$: there is $\gamma \in \Gamma$ such that $\pi_\gamma(x) = \pi_\gamma(y)$ for any $x, y \in W$ and any $\beta \in \Gamma - \{\gamma\}\}$. An element $U \in Z$ is said to be Σ-coherent if it contains M and it is closed under Q-joins. We may define a map $s : Z \to Z$ by $C \mapsto \bigwedge \{U \geqslant C : U$ is Σ-coherent$\}$. Evidently, s is a nucleus on Z and we denote by L the sublocale associated with s. Moreover, we may define continuous map $p_\gamma : L \to L_\gamma$ by $\tau(p_\gamma(x)) = s(\downarrow x'_\gamma) = s(x'_\gamma)$, where $\pi_\gamma(x'_\gamma) = x$, $\pi_\beta(x'_\gamma) = 1$ for $\beta \neq \gamma$.

Now, we have the following result.

Proposition 1.1. *L is the product of the locales L_γ with projections p_γ.*

Consider a system S_γ of covers in L_γ such that $S_\gamma = \{1\}$ except for, at most, finitely many $\gamma \in \Gamma$. Then the system $\mathbb{X}S_\gamma = \{\bigwedge_{\gamma \in \Gamma} s(x(\gamma)'_\gamma) : x(\gamma) \in S_\gamma, \gamma \in \Gamma\}$ is a cover of L.

Recall the following important lemma due to Kříž [6].

Lemma 1.2. *Let $T = \{s(t) : t \in T'\}$, $T' \subseteq B$ be a finite cover of L. Then there is a system S_γ of covers in L_γ such that $S_\gamma = \{1\}$ except for, at most, finitely many $\gamma \in \Gamma$ and $\mathbb{X}S_\gamma < T$.*

2. Finitely regular locales

Definition 2.1. Let L be a locale, $C, D \subseteq \tau(L)$. We shall say that D *finitely regular* (*strongly*) *refines* C if, for each $d \in D$, there exists $F \subseteq\subseteq C$ $(c \in C)$ such that $d \lhd \bigvee F$ $(d \lhd c)$. We shall write $D \lhd_f C$ $(D \lhd C)$. Clearly, $\lhd \subseteq \lhd_f$. Putting $\sigma(C) = \bigcup \{X \subseteq L: X \lhd_f C\}$ we have that $\sigma(C) \lhd_f C$.

Recall that the relations $\lhd_f$, $\lhd$ are preserved by frame morphisms and that the following holds:

(i) $E < D \lhd_f C < F$ implies $E \lhd_f F$,

(ii) $E < D \lhd C < F$ implies $E \lhd F$,

(iii) $D_i \lhd_f C_i$, $i \in I$, $I \neq \emptyset$ implies $\bigcup D_i \lhd_f \bigcup C_i$,

(iv) $D_i \lhd C_i$, $i \in I$, $I \neq \emptyset$ implies $\bigcup D_i \lhd \bigcup C_i$,

(v) $B \lhd_f A$, $D \lhd_f C$ implies $B \wedge D \lhd_f A \wedge C$,

(vi) $B \lhd A$, $D \lhd C$ implies $B \wedge D \lhd A \wedge C$.

We shall say that a locale L is *finitely* (*cover*) *regular* if each cover of L has a finitely regular (strong) refinement which is a cover as well. Regular locales are cover regular and cover regular locales are finitely regular.

The following proposition was our basic motivation for the investigation of finitely regular locales.

Proposition 2.2. *Let L be a paracompact locale. Then L is finitely regular.*

Proof. Let $C \in \mathscr{C}(L)$. Then there is a locally finite cover D, $D < C$, i.e., there is $E \in \mathscr{C}(L)$ such that $D(e) = \{d \in D: d \wedge e \neq 0\}$ is finite for each $e \in E$. Let $e \in E$. Then

$$1 = \bigvee D(e) \vee \bigvee (D - D(e)) \leq \bigvee D(e) \vee e^*$$

i.e., $E \lhd_f D < C$. $\square$

Corollary 2.3. *Every compact locale is finitely regular.*

Theorem 2.4. *Let L be a Lindelöf locale. Then L is paracompact if and only if L is finitely regular.*

Proof. ($\Rightarrow$) By 2.2.

($\Leftarrow$) Let C be a cover of L. Then there is $D \in \mathscr{C}(L)$ such that $D \lhd_f C$. If D is finite we are ready. Otherwise we can find a countable cover $E = \{e_n: n \in \mathbb{N}\} \subseteq D$. Evidently, for each $e \in E$ there is $C_e \subseteq\subseteq C$, such that $e \lhd \bigvee C_e$. We put $C' = \bigcup \{C_e: e \in E\}$. Then C' is at most countable, C' is a cover in L. If C' is finite we are ready. Let $C' = \{c_n: n \in \mathbb{N}\}$. Then we may define an increasing function $f: \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ such that

$$f(0) = 0, \qquad f(1) = 1 \quad \text{and} \quad \bigvee_{i=1}^{f(n)} e_i \lhd \bigvee_{j=1}^{f(n+1)} c_j.$$

By induction, we then define elements $v_{n,j}$, w_j, $j \in \mathbb{N}$, $n \in \mathbb{N} \cup \{0\}$ as follows:

$$v_{0,j} = c_j, \tag{1}$$

$$v_{n+1,j} = \begin{cases} c_j \wedge \left(\bigvee_{i=1}^{f(n)} e_i\right) & \text{for } j > f(n+1), \\ v_{n,j} & \text{otherwise}, \end{cases} \tag{2}$$

$$w_j = v_{n,j} \quad \text{for } f(n) < j \leq f(n+1). \tag{3}$$

Evidently, $W = \{w_j : j \in \mathbb{N}\} < C$ and $\bigvee_{i=1}^{f(n)} w_i = \bigvee_{i=1}^{f(n)} c_i$. By an obvious induction we have

$$\begin{aligned} \bigvee_{i=1}^{f(n+1)} w_i &= \bigvee_{i=1}^{f(n)} w_i \vee \bigvee_{i=f(n)+1}^{f(n)} w_i = \bigvee_{i=1}^{f(n)} c_i \vee \bigvee_{i=f(n)+1}^{f(n)} c_i \wedge \left(\bigvee_{k=1}^{f(n-1)} e_k\right) \\ &= \bigvee_{i=1}^{f(n)} c_i \vee \bigvee_{i=f(n)+1}^{f(n)} c_i \wedge \left(\bigvee_{j=1}^{f(n)} c_j \vee \left(\bigvee_{k=1}^{f(n-1)} e_k\right)^*\right) = \bigvee_{i=1}^{f(n+1)} c_i. \end{aligned}$$

Now, W is a countable cover and for each $e_i \in E$ we have

$$e_i \wedge w_j \neq 0 \text{ implies } j \leq f(n+1), f(n) < i.$$

Then W is locally finite. $\square$

Note that any codense sublocale, finitary sublocale respectively of a finitely regular locale is finitely regular as well and that for a finitely regular locale all its sublocales are finitely regular if and only if all its open sublocales are finitely regular.

Lemma 2.5 (AC). *Let X be a topological space. Then the following conditions are equivalent*:

(i) *X is finitely regular.*

(ii) *There is a subbase $\mathcal{S}$ of X such that each cover $\mathcal{U} \subseteq \mathcal{S}$ is finitely regular.*

Proof. (i)$\Rightarrow$(ii). It is trivial.

(ii)$\Rightarrow$(i). Let X be not finitely regular. Then there is an open cover $\mathcal{C}$ of X that is not finitely regular, i.e., there is an element $x \in X$, $x \notin \bigcup \sigma(\mathcal{C})$. We put $\mathcal{B} = \{\mathcal{V}: \mathcal{V}$ is an open cover of X, $x \notin \bigcup \sigma(\mathcal{V})\}$. Evidently, $\mathcal{B}$ is closed under any union of chains in $\mathcal{B}$, $\mathcal{B} \neq \emptyset$, i.e., there is a maximal element $\mathcal{U}$ in $\mathcal{B}$. We denote by $\mathcal{A} = \{A \in \tau(X): A \notin \mathcal{U}\}$. One can easily check that $\mathcal{A}$ is a filter on $\tau(X)$.

Let $z \in X$. Since $\mathcal{U}$ is a cover of X there is $U \in \mathcal{U}$, $z \in U$. By the subbase property there are $S_1, \ldots, S_k \in \mathcal{S}$ such that $z \in \bigcap S_j \notin \mathcal{A}$, i.e., there is $1 \leq j \leq k$ such that $S_j \notin \mathcal{A}$. Therefore $z \in S_j \in \mathcal{U} \cap \mathcal{S}$. Since z was arbitrary we have that $\mathcal{U} \cap \mathcal{S}$ is an open cover of $X, x \in X = \bigcup \sigma(\mathcal{U} \cap \mathcal{S}) \subseteq \bigcup \sigma(\mathcal{U})$, a contradiction. $\square$

It follows from the proof that Lemma 2.5 holds even if we replace X by a locale L such that every of its elements ($\neq 1$) is contained in a prime one.

By a standard applicaton of Lemma 2.5 we have the following.

Theorem 2.6 (AC). *The product of finitely regular spaces is finitely regular.*

Another proof of Theorem 2.6 using nets may be found in [5].

Remark 2.7. We shall define a function $\Theta : \mathcal{Set} \times \mathcal{Set} \to \mathcal{CompTop}_1$ by $(A, B) \mapsto (X, \tau(X))$, where $X = A \cup B$, $\emptyset$, $B \in \tau(X)$ and a complement of any finite subset is contained in $\tau(X)$, $\tau(X)$ is minimal with this property. We put $B^0 = X$, $B^1 = B$. Then $U \in \tau(X) \Leftrightarrow U = B^\alpha \cap C$, $\alpha \in \{0, 1\}$, $X - C$ is finite or X. Evidently, X is T_1. It is easy to verify that X is a compact space.

By a straightforward computation we have the following.

Lemma 2.8. *Let A, B be sets, B infinite, $X = \Theta(A, B)$. Then* $\mathrm{int}(C) \neq \emptyset$ *implies* $\mathrm{cl}(C) = X$ *for each* $C \subseteq X$.

Theorem 2.9. *The following two conditions are equivalent*:

(i) AC.

(ii) *The product of finitely regular spaces is finitely regular.*

Proof. (i)$\Rightarrow$(ii). By 2.6.

(ii)$\Rightarrow$(i). Let X_γ be non-empty, $\gamma \in \Gamma \neq \emptyset$. Then there is an infinite set B such that $B \cap \bigcup \{X_\gamma : \gamma \in \Gamma\} = \emptyset$. We put $Y_\gamma = \Theta(X_\gamma, B)$, $Y = \prod Y_\gamma$, $X = \prod X_\gamma$, $U_\gamma = p_\gamma^{-1}(X_\gamma)$, $Z_\gamma = p_\gamma^{-1}(B)$. Then Y is finitely regular. Let $\mathcal{I}$ be an ideal generated by Z_γ, $\gamma \in \Gamma$. One can easily check that $Y \neq \bigcup \mathcal{I}$, i.e., X is non-empty. $\square$

3. A general cover theorem

This section contains the central results of the paper. It unifies the results concerning products of locales obtained by Isbell [3], Dowker and Strauss [1], Johnstone [4], Kříž [6].

We shall show that under some restricted assumptions certain covering properties of locales are preserved in products. Especially, the validity of the General cover theorem is given. Axiom of choice is prerequisite of our considerations.

In the sequel, the $\mathcal{B}$, $\mathcal{F}$ be functions with the domain $\mathcal{Loc}$ and the codomain $\mathcal{Set}$ satisfying the following conditions:

(A0) $L \in \mathcal{Loc}$ implies $1 \in \mathcal{B}(L)$, $\mathcal{B}(L) \subseteq \tau(L)$ is closed under finite meets,

(A1) $L \in \mathcal{Loc}$, $t \in \mathcal{B}(L)$ implies $\{t\} \in \mathcal{F}(L)$, $\mathcal{F}(L) \subseteq 2^{\mathcal{B}(L)}$,

(A2) $L \in \mathcal{Loc}$, $t \in \mathcal{B}(L)$, $F \in \mathcal{F}(L)$ implies $\{t\} \wedge F \in \mathcal{F}(L)$, (A3) $L \in \mathscr{L}oc$, $F \in \mathscr{F}(L)$ and for each $f \in \mathscr{F}(L)$ there is $F_f \in \mathscr{F}(L)$ such that $f = \bigvee F_f$ implies $\bigcup \{F_f : f \in F\} \in \mathscr{F}(L)$,

(A4) $L, K \in \mathscr{L}oc$, $F \in \mathscr{F}(L)$, $H \in \mathscr{F}(K)$ implies $\tau(\pi_L)(F) \wedge \tau(\pi_K)(H) \in \mathscr{F}(L \times K)$.

If $\tau(h)(\mathscr{F}(L)) \subseteq \mathscr{F}(L')$ for each epimorphism $h: L' \to L$ in $\mathscr{L}oc$ then (A4) is a consequence of (A1), (A2) and (A3). If $F \in \mathscr{F}(L)$, we shall speak about a set of a type $\mathscr{F}$. Recall that the foregoing conditions (A1), (A2) and (A3) in fact coincide with the conditions (i), (ii) and (iii) in [1]. It is sometimes convenient to suppose the following strengthening of the condition (A1):

(A1′) $L \in \mathscr{L}oc$, $C \subseteq\subseteq \mathscr{B}(L)$ implies $C \in \mathscr{F}(L)$, $\mathscr{F}(L) \subseteq 2^{\mathscr{B}(L)}$.

Definition 3.1. We shall say that a locale L is an *$\mathscr{F}$-locale* if $C \in \mathscr{C}(L)$ implies there is $D \in \mathscr{C}(L) \cap \mathscr{F}(L)$ such that $D < C$, i.e., every cover of L has a refinement of the type $\mathscr{F}$.

A morphism $f: L \to K$ of locales is said to be $\mathscr{F}$-good if $\tau(f)(\mathscr{F}(K)) \subseteq \mathscr{F}(L)$. We shall say that a space X is an *$\mathscr{F}$-space* if $\tau(X)$ is an $\mathscr{F}$-locale.

In the following, we shall work with the notation introduced in Section 1.

Remark 3.2. In the sequel, let L_γ, $\gamma \in \Gamma$, be a family of finitely regular $\mathscr{F}$-locales, $K \subseteq B$, $\bigvee s(K) = 1$,

$$\begin{aligned}\bar{K} = \{x \in B : x \in M' \subseteq\subseteq B,\ \bigvee s(M') = 1 \text{ implies there is a subset } Q'_x \text{ of } B,\\ s(Q'_x) < s(K \cup (M' - \{x\})),\ s(Q'_x) \lhd_f s(K \cup (M' - \{x\})),\\ s(Q'_x) \in \mathscr{F}(L) \cap \mathscr{C}(L)\}.\end{aligned}$$

Lemma 3.3. *Let S_γ, $\gamma \in \Gamma$ be a system of covers of L_γ, $\gamma \in \Gamma$, $S_\gamma = \{1\}$ for all but finitely many $\gamma \in \Gamma$. Then there are T_γ, $\gamma \in \Gamma$, $T_\gamma < S_\gamma$, $T_\gamma \lhd_f S_\gamma$, $T_\gamma \in \mathscr{C}(L_\gamma) \cap \mathscr{F}(L_\gamma)$, $\gamma \in \Gamma$ such that $T_\gamma = \{1\}$ for all but finitely many $\gamma \in \Gamma$.*

Proof. Let $\Gamma_0 \subseteq \Gamma$ be finite such that $\gamma \notin \Gamma_0$ implies $S_\gamma = \{1\}$. For each $\gamma \in \Gamma_0$ there is T_γ, $T_\gamma < S_\gamma$, $T_\gamma \lhd_f S_\gamma$, $T_\gamma \in \mathscr{C}(L_\gamma) \cap \mathscr{F}(L_\gamma)$, for each $\gamma \notin \Gamma_0$ we put $T_\gamma = \{1\}$, i.e., $T_\gamma < S_\gamma$, $T_\gamma \lhd_f S_\gamma$, $T_\gamma \in \mathscr{C}(L_\gamma) \cap \mathscr{F}(L_\gamma)$ by (A0), (A1). $\square$

Lemma 3.4. *Let $t \in B$, $s(t) \in \mathscr{B}(L)$, $A \subseteq\subseteq B$, $W \subseteq\subseteq \bar{K}$, $(K \cup A) \cap W = \emptyset$, $s(t) \lhd_f (A \cup W)$. Then there is $T \subseteq B$ such that $s(T) < s(K \cup A)$, $s(T) \lhd_f s(K \cup A)$, $s(T) \in \mathscr{F}(L)$, $\bigvee s(T) = s(t)$.*

Proof. The proof will be done by induction according to card W. If card $W = 0$ we are ready—we put $T = \{t\}$. Let card $W = n > 0$ and the lemma holds for all $k < n$. Then there is an element $w \in W$ such that

$$s(D(t) \cup A \cup (W - \{w\}) \cup \{w\}) = 1.$$

Since $w \in \bar{K}$, $D(t) \cup A \cup (W - \{w\})$ is finite there is $Q_w \subseteq B$,

$$s(Q_w) < s(K \cup (D(t) \cup A \cup (W - \{w\})),$$
$$s(Q_w) \lhd_f s(K \cup D(t) \cup A \cup (W - \{w\})),$$
$$s(Q_w) \in \mathscr{F}(L) \cap \mathscr{C}(L)).$$

We put $S = \{t\} \wedge Q_w$. Then $s(S) < s(K \cup A \cup (W - \{w\}))$. Let $z \in S$. Then $z = t \wedge y$ for some $y \in Q_w$. We have

$$1 = s(y)^* \vee s(K' \cup D(t) \cup A \cup (W - \{w\})), \quad K' \subseteq\subseteq K.$$

This implies $1 = s(z)^* \vee s(K' \cup A \cup (W - \{w\}))$, i.e.,

$$s(z) \lhd_f s(K \cup A \cup (W - \{w\})), \qquad \bigvee s(S) = s(t), \qquad s(S) \in \mathscr{F}(L)$$

by (A2). By induction, for each $z \in S$ there is $T_z \subseteq B$ such that

$$s(T_z) < s(K \cup A), \qquad s(T_z) \lhd_f s(K \cup A), \qquad \bigvee s(T_z) = s(z),$$
$$s(T_z) \in \mathscr{F}(L).$$

We put $T = \bigcup \{T_z : z \in S\}$. Then by (A3) $s(T) \in \mathscr{F}(L)$,

$$\bigvee s(T) = \bigvee \{T_z : z \in S\} = \bigvee s(S) = s(t), s(T) \lhd_f s(K \cup A). \quad \square$$

Proposition 3.5. $\bar{K} = B$.

Proof. Evidently, $K \subseteq \bar{K}$. Namely, if $k \in K$, $k \in M' \subseteq\subseteq B$, $\bigvee s(M') = 1$ then by Lemma 1.2 there are finite covers, S_γ, $\gamma \in \Gamma$ of L_γ, $\gamma \in \Gamma$, $S_\gamma = \{1\}$ for all but finitely many $\gamma \in \Gamma$, $\mathbb{X}S_\gamma < s(M')$. Then by Lemma 3.3 there are T_γ, $\gamma \in \Gamma$, $T_\gamma < S_\gamma$, $T_\gamma \lhd_f S_\gamma$, $T_\gamma \in \mathscr{C}(L_\gamma) \cap \mathscr{F}(L_\gamma)$, $\gamma \in \Gamma$ such that $T_\gamma = \{1\}$ for all but finitely many $\gamma \in \Gamma$. We put $Q_k = \mathbb{X}T_\gamma$. Then $Q_k = s(Q'_k)$, $Q'_k \subseteq B$, and by (A4) and that $M' \subseteq\subseteq K \cup (M' - \{k\})$ we have that

$$Q_k < s(K \cup (M' - \{k\})), \qquad Q_k \lhd_f s(K \cup (M' - \{k\})),$$
$$Q_k \in \mathscr{F}(L) \cap \mathscr{C}(L).$$

It is enough to check that $\bar{K} \in L$ (then $1 = \bigvee s(K) \in \bar{K}$). We shall show that $\bar{K} \in \downarrow B$. Let $x \in \bar{K}$, $z \leqslant x$, $z \in B$, $z \in M' \subseteq\subseteq B$, $\bigvee s(M') = 1$. Then

$$\bigvee (s(M' - \{z\}) \cup \{x\}) = 1$$

and implies there is a subset Q'_x of B,

$$s(Q'_x) < s(K \cup (M' - \{z\})), \qquad s(Q'_x) \lhd_f s(K \cup (M' - \{z\})),$$
$$s(Q'_x) \in \mathscr{F}(L) \cap \mathscr{C}(L)\}.$$

We may put $Q'_z = Q'_x$. Then $z \in \bar{K}$.

We shall verify that the conditions of the construction of a product in locales from Section 1 are satisfied.

(i) Let $x \in M$, i.e., $s(x) = 0$, $x \in M' \subseteq\subseteq B$, $\bigvee s(M') = 1$. Then $\bigvee s(M' - \{x\}) = 1$. By Lemma 1.2 there are finite covers S_γ, $\gamma \in \Gamma$ of L_γ, $\gamma \in \Gamma$, $S_\gamma = \{1\}$ for all but finitely many $\gamma \in \Gamma$, $\mathbb{X}S_\gamma < s(M' - \{x\})$. Then by Lemma 3.3 there are T_γ, $\gamma \in \Gamma$, $T_\gamma < S_\gamma$, $T_\gamma \lhd_f S_\gamma$, $T_\gamma \in \mathscr{C}(L_\gamma) \cap \mathscr{F}(L_\gamma)$, $\gamma \in \Gamma$ such that $T_\gamma = \{1\}$ for all but finitely many $\gamma \in \Gamma$. We put $Q_x = \mathbb{X}T_\gamma$. Then

$$Q_x < s(K \cup (M' - \{x\})), \qquad Q_x \lhd_f s(K \cup (M' - \{x\})),$$
$$Q_x \in \mathscr{F}(L) \cap \mathscr{C}(L).$$

Then $x \in \bar{K}$, i.e., $M \subseteq \bar{K}$.

(ii) Let $W \subseteq \bar{K}$ and there is an index $\gamma_0 \in \Gamma$ such that for all $v, w \in W$ there is $\pi_\gamma(v) = \pi_\gamma(w)$ for all $\gamma \in \Gamma - \{\gamma_0\}$. We put $x = \bigvee W$. Let $x \in M' \subseteq\subseteq B$, $\bigvee s(M') = 1$. Again by Lemma 1.2 we have that there are finite covers S_γ, $\gamma \in \Gamma$ of L_γ, $\gamma \in \Gamma$, $S_\gamma = \{1\}$ for all but finitely many $\gamma \in \Gamma$ and $\mathbb{X}S_\gamma < s(M')$. We put

$$S'_{\gamma_0} = \{y \in S_{\gamma_0} : y \nleq \pi_{\gamma_0}(x)\} \cup \{\pi_{\gamma_0}(w) : w \in W\},$$

$S'_\gamma = S_\gamma$ for $\gamma \neq \gamma_0$. By Lemma 3.3 there are T_γ, $\gamma \in \Gamma$, $T_\gamma < S'_\gamma$, $T_\gamma \lhd_f S'_\gamma$, $T_\gamma \in \mathscr{C}(L_\gamma) \cap \mathscr{F}(L_\gamma)$, $\gamma \in \Gamma$ such that $T_\gamma = \{1\}$ for all but finitely many $\gamma \in \Gamma$. We put $T = \mathbb{X}T_\gamma$, $S' = \mathbb{X}S'_\gamma$. Then

$$T < S' < s((M' - \{x\}) \cup W), \qquad \bigvee T = 1$$

and clearly $T \lhd_f s(S')$, $T \in \mathscr{F}(L)$ by (A4).

Evidently, for each $s(t) \in T$ there are

$$A_t \subseteq\subseteq (M' - \{x\} \cup (W \cap K) \subseteq (M' - \{x\} \cup K, W_t$$
$$\subseteq\subseteq (W - K) \cap (B - (M' - \{x\})) \subseteq \bar{K}$$

such that $(A_t \cup K) \cap W_t = \emptyset$ and $s(t) \lhd_f s(A_t \cup W_t)$. By the preceding lemma, there are sets $T_t \subseteq B$ such that

$$s(T_t) < s(K \cup A_t), \qquad s(T_t) \lhd_f s(K \cup A_t), \qquad s(T_t) \in \mathscr{F}(L),$$
$$\bigvee s(T_t) = s(t).$$

We put $Q'_x = \bigcup \{T_t : t \in T\}$. Then $Q'_x \subseteq B$,

$$s(Q'_x) < s(K \cup (M' - \{x\})), \qquad s(Q'_x) \lhd_f s(K \cup (M' - \{x\})),$$
$$s(Q'_x) \in \mathscr{F}(L) \cap \mathscr{C}(L)\}$$

by (A3). We have that $x \in \bar{K}$. □

Theorem 3.6 (General cover theorem). *Let L_γ, $\gamma \in \Gamma$ be a family of finitely regular $\mathscr{F}$-locales. Then the product $L = \Pi\{L_\gamma, \gamma \in \Gamma\}$ is a finitely regular $\mathscr{F}$-locale.*

Proof. Let $C \in \mathscr{C}(L)$. Then there is a basic refinement K of C. By Proposition 3.5 $1 \in K$, i.e., putting $M' = \{1\}$ there is a subset Q of B, $s(Q) < s(K)$, $s(Q) \lhd_f s(K)$, $s(Q) \in \mathscr{F}(L) \cap \mathscr{C}(L)\}$. □

Corollary 3.7. *Let X_γ, $\gamma \in \Gamma$ be a family of finitely regular $\mathscr{F}$-spaces, $f: \Omega(\Pi X_\gamma) \to \Pi\Omega(X_\gamma)$ be an induced morphism of locales which is $\mathscr{F}$-good. Then we have*

(i) *If f is an isomorphism or a codense map then ΠX_γ is a finitely regular $\mathscr{F}$-space.*

(ii) *If Π is countable and X_γ, $\gamma \in \Gamma$ are Čech complete spaces then ΠX_γ is a finitely regular $\mathscr{F}$-space.*

(iii) *If Γ is finite and X_γ, $\gamma \in \Gamma$ are locally compact spaces for all but at most one $\gamma \in \Gamma$ then ΠX_γ is a finite regular $\mathscr{F}$-space.*

Let $\mathscr{F}$ satisfies (A1′). *Then*

(iv) *If f is a finitary map then ΠX_γ is a finitely regular $\mathscr{F}$-space.*

Proof. (i) Trivial.

(ii), (iii) follows from $\Omega(\Pi X_\gamma) = \Pi\Omega(X_\gamma)$ (see [2, 4]).

(iv) Let $C \in \mathscr{C}(\Omega(\Pi X_\gamma))$, C be basic. Then we put $C' = \{\bigvee F : F \subseteq\subseteq C\}$, $\bigvee$ is computed in $\Omega(\Pi X_\gamma)$. Clearly, $C' \subseteq \tau(\Pi\Omega(X_\gamma))$ is directed and $\tau(f)(C') \in \mathscr{C}(\Omega(\Pi X_\gamma))$. Using finitarity we have that $C' \in \mathscr{C}(\Pi\Omega(X_\gamma))$. Then there is a basic refinement $s(Q')$ of C', i.e., for each $q \in Q'$ there is a finite set $F(q) \subseteq C$ such that $s(q) \leq \bigvee F(q)$, $s(Q') \lhd_f s(C')$,

$$s(Q') \in \mathscr{F}(\Pi\Omega(X_\gamma)) \cap \mathscr{C}(\Pi\Omega(X_\gamma))\}.$$

We put $Q = \{q \wedge F(q) : q \in Q'\}$. By (A1′) and (A3) we have

$$\tau(f)(s(Q)) \in \mathscr{F}(\Omega(\Pi X_\gamma)) \cap \mathscr{C}(\Omega(\Pi X_\gamma))\}, \qquad \tau(f)(s(Q)) < C. \qquad \square$$

Corollary 3.8. *Let L_γ, $\gamma \in \Gamma$ be a family of finitely regular locales. Then the product L is a finitely regular locale.*

Proof. We put $\mathscr{B}(L) = L$, $\mathscr{F}(L) = 2^L$. Clearly, the conditions (A0)–(A4) are satisfied. $\square$

4. Applications

In this section we use the general cover theorem to investigate the behaviour of some covering properties in localic and topologicl products. As a special result we obtain the following well-known theorem (see [2, 4, 6]).

Theorem 4.1 (Tychonoff product theorem for locales). *Let L_γ, $\gamma \in \Gamma$ be a family of compact locales. Then the product L is compact.*

Proof. We put $\mathscr{B}(L) = L$, $\mathscr{F}(L) = \mathscr{F}in(L) = \{A : A \subseteq\subseteq \tau(L)\}$. One can easily check that (A0)–(A1) are satisfied. $\square$

Corollary 4.2 (Tychonoff product theorem). *Let X_γ, $\gamma \in \Gamma$ be a family of compact spaces. Then the product $X = \Pi\{X_\gamma : \gamma \in \Gamma\}$ is compact.*

Proof. Since every compact locale has a maximal element we have that the map f from Corollary 3.7 is codense. Using the 'goodness' of f we get the proposition. □

The following extends the results of Dowker and Strauss' paper [2].

Theorem 4.3. *Let L_γ, $\gamma \in \Gamma$ be a family of paracompact locales. Then the product L is paracompact.*

Proof. We put $\mathscr{B}(L) = L$, $\mathscr{F}(L) = \mathscr{L}\mathscr{F}\!in(L) = \{A : A \subseteq \tau(L),\ A$ is locally finite$\}$. Since every paracompact locale is finitely regular the rest follows from Theorem 4.1. □

Corollary 4.4. *Let X, Y be paracompact spaces, X locally compact. Then the product $X \times Y$ is paracompact.*

According to Theorem 4.3 and Corollary 4.4, we may substitute paracompactness by prime-metacompactness and adding finite regularity we obtain the following.

Theorem 4.5. *Let L_γ, $\gamma \in \Gamma$ be a family of prime-metacompact finitely regular locales. Then the product L is prime-metacompact.*

Corollary 4.6. *Let X, Y be metacompact finitely regular spaces, X locally compact. Then the product $X \times Y$ is metacompact.*

Recall that a locale is ***c**-compact,* $\boldsymbol{c}$ being a cardinal, if every cover has subcover of a cardinality $\leqslant \boldsymbol{c}$. Proceeding in the same vein as above we have the following.

Theorem 4.7 (AC). *Let L_γ, $\gamma \in \Gamma$ be a family of **c**-compact finitely regular locales. Then the product L is **c**-compact.*

Corollary 4.8 (AC). *Let X, Y be **c**-compact finitely regular spaces, X locally compact. Then the product $X \times Y$ is **c**-compact.*

Corollary 4.9 (AC). *Let X_γ, $\gamma \in \Gamma$ be a countable family of **c**-compact Čech-complete spaces. Then the product X is **c**-compact.*

Since (AC) is needed to verify (A3) using its equivalent $\boldsymbol{c} \cdot \boldsymbol{c} = \boldsymbol{c}$ we may avoid it (for $\boldsymbol{c} = \aleph_0$) as follows.

Theorem 4.10. *Let L_γ, $\gamma \in \Gamma$ be a family of Lindelöf finitely regular locales. Then the product L is Lindelöf.*

Corollary 4.11. *Let X, Y be Lindelöf finitely regular spaces, X locally compact. Then the product $X \times Y$ is Lindelöf.*

We shall close this section with the case of screenable and supercompact locales. A locale is said to be *screenable, supercompact* if every cover has a σ-disjoint, one element refinement respectively.

Theorem 4.12. *Let L_γ, $\gamma \in \Gamma$ be a family of screenable finitely regular locales. Then the product L is screenable.*

Corollary 4.13. *Let X, Y be screenable finitely regular spaces, X locally compact. Then the product $X \times Y$ is screenable.*

Corollary 4.14. *Let X_γ, $\gamma \in \Gamma$ be a countable family of screenable Čech-complete spaces. Then the product $X = \Pi\{X_\gamma \colon \gamma \in \Gamma\}$ is screenable.*

Theorem 4.15. *Let L_γ, $\gamma \in \Gamma$ be a family of supercompact locales. Then the product L is supercompact.*

Proof. We put $\mathscr{B}(L) = \{1\}$, $\mathscr{F}(L) = \{\{1\}\}$. $\square$

Corollary 4.16. *Let X_γ, $\gamma \in \Gamma$ be a family of supercompact spaces. Then the product $X = \Pi\{X_\gamma \colon \gamma \in \Gamma\}$ is supercompact.*

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